OPTIMUM CONTROL OF CONDUCTIVITY OF A FLUID MOVING IN A CHANNEL IN A MAGNETIC FIELD

(OPTIMAL'NOE UPRAVLENIE PROVODIMOST'IU ZHIDKOSTI, DVIZHUSHCHEISIA PO KANALU V MAGNITNOM POLE)

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The problem of determination of currents and electric fields in a slightly conducting fluid moving in a flat channel in a perpendicular magnetic field has been examined by a number of authors [1 to 4]; the most comprehensive of these studies belong to A.B.Vatazhin.

In all references mentioned, calculations of fields and currents were carried out under the assumption that the electric conductivity of the fluid, the distribution of velocity, and the external magnetic field (to all of which we will refer subsequently as controls), are somehow or other given functions of coordinates. In those cases where it was possible to obtain a solution for controls which were prescribed arbitrarily in a certain class of functions (for example, for constant conductivity and constant magnetic field, while the velocity depended only on the transverse coordinate) it was permissible to select such functions from the indicated classes which corresponded to optimum regimes in a definite sense.

However, general solutions of this type can be obtained in quite rare cases. On the other hand, if the problem of optimization (in a definite sense) is presented from the very beginning for the distribution of currents with respect to controls which can be selected from functions of some (if possible, sufficiently broad) class, then for the solution of this problem the knowledge of distribution of currents for arbitrary control in the indicated class is not required at all *. A general method of solution for this type of optimum problems in mathematical physics was developed in [5]. In this reference a procedure is indicated which allows, from the very beginning, to isolate controls which can only turn out to be optimum (Weierstrass condition). Further investigation is confined only to selected controls and in many cases can be carried to conclusion.

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* If, however, such a solution is known, then the search for optimum controls is simplified in a corresponding fashion: the problem of Mayer-Bolza of variational calculus reduces to the simplest case. Besides, it is understood that no special methods are required for the determination of the optimum control.

The present study contains an example for the application of the general theory developed in [5]. The problem of finding the optimum control $\sigma(x,y)$ among sectionally continuous functions $\sigma(x,y)$ (conductivity of the fluid) of two independent variables is examined. The functions satisfy the inequality $\sigma_{min} \leqslant \sigma(x,y) \leqslant \sigma_{max}$. The optimum control $\sigma(x,y)$ corresponds to a distribution of currents and to a distribution of electric field which satisfy certain experimental requirements. These requirements, together with conditions for the problem, are formulated in detail in Section 1.

The solution of the problems turns out to be sufficiently simple; this is of course connected with the fact that the control $\sigma(x,y)$ enters linearly into the original equations. Because of this the inequality, limiting possible values $\sigma(x,y)$, plays a decisive role in the determination of the optimum control.

1. The distribution of current j and potential of electric field z^1 in a conducting medium moving with the velocity $\mathbf{v}(v(x,y),0,0)$ in a magnetic field $\mathbf{B}(0,0,-B(x))$ is described by Equations [1]

$$\operatorname{div} \mathbf{j} = 0, \ \mathbf{j} = \sigma\cdot(-\operatorname{grad} z^1 + \frac{1}{c}\ [\mathbf{v},\ \mathbf{B}]) \qquad (1.1)$$

Here σ is the electric conductivity of the fluid.

We will consider functions $v(y)$ and $B(x)$ as given. As is well known, this corresponds to the frequently used in magnetohydrodynamics approximation of small magnetic Reynolds numbers when the distribution of velocities practically coincides with the hydrodynamic distribution and the induced magnetic fields are negligibly small *.

As for the function $\sigma(x,y)$, its values are determined at any point in the stream by the possibility at our disposal of controlling the conductivity of the fluid. As a rule those possibilities are limited and in the best case one succeeds in reaching some maximum value of conductivity σ_{max}. On the other hand, the conductivity of the fluid itself (in the absence of external interactions such as heating **, additives, etc.) determines the minimum possible value σ_{min}. Therefore, it can be assumed that the conductivity in all cases satisfies the inequality

$$\sigma_{\min} \leqslant \sigma\ (x,\ y) \leqslant \sigma_{\max} \qquad (1.2)$$

This inequality is quite essential for the following treatment.

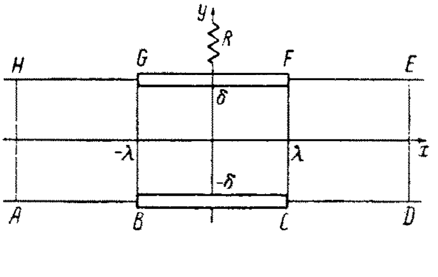

Fig.1

Introducing the notation

$$\mathbf{j} = -\operatorname{curl}(\mathbf{k}z^2), \qquad j_x = \zeta^1, \quad j_y = \zeta^2, \qquad \rho = 1/\sigma \qquad (1.3)$$

* Functions $v(y)$ and $B(x)$ are fixed only for the purpose of simplifying the optimum problem. It is possible of course to optimize the distribution of currents with respect to these controls also.

** We neglect the temperature dependent change in conductivity due to Joule heating.

Equations (1.1) and inequality (1.2) are written in the following form

$$\frac{\partial z^1}{\partial x} = -\rho\zeta^1, \qquad \frac{\partial z^1}{\partial y} = -\rho\zeta^2 + \frac{vB}{c}, \qquad \frac{\partial}{\partial x}\left(\frac{vB}{c} - \rho\zeta^2\right) + \frac{\partial}{\partial y}\rho\zeta^1 = 0$$
$$\frac{\partial z^2}{\partial x} = \zeta^2, \qquad \frac{\partial z^2}{\partial y} = -\zeta^1, \qquad \frac{\partial\zeta^1}{\partial x} + \frac{\partial\zeta^2}{\partial y} = 0 \qquad (1.4)$$

$$\rho_{\min} \leqslant \rho(x, y) \leqslant \rho_{\max}$$

The boundary conditions of the problem can be most diversified. We will examine the case where the walls $y = \pm\delta$ of the channel (Fig. 1) are insulators everywhere with exception of regions $|x| < \lambda$, which are located on both walls opposite each other and which represent ideally conducting electrodes [1]. The latter are connected through the load R, through which electrical current flows when fluid moves in the magnetic field.

$$I = \int_{-\lambda}^{\lambda} \zeta^2(x, \pm\delta)\, dx \qquad (1.5)$$

We also present the following expression for the magnitude of Joule losses

$$\dot{Q} = \int_{-\delta}^{\delta} dy \int_{-\infty}^{\infty} dx\, [(\zeta^1)^2 + (\zeta^2)^2]\, \rho(x, y) \qquad (1.6)$$

For the schematic shown in Fig.1, the problem of conductivity control of the field will be solved in such a manner that (Problem 1) the functional I will reach the maximum possible value or (Problem 2) the functional Q will reach the minimum possible value.

The boundary conditions for the problems presented will be spelled out in Section 4.

2. Equations for Lagrange's multipliers. Problem 1. According to [5]

$$H^{(1)} = -\xi_1\rho\zeta^1 + \xi_2\zeta^2 + \eta_1\left(\frac{vB}{c} - \rho\zeta^2\right) - \eta_2\zeta^1 -$$
$$- \Gamma^* [(\rho_{\max} - \rho)(\rho - \rho_{\min}) - \rho_*^2] \qquad (2.1)$$

Here ρ_* is an additional control and ξ_1, η_1, Γ^* are Lagrange's multipliers *. The conditions for stationary state are reduced to Equations

$$\partial\xi_1 / \partial x + \partial\eta_1 / \partial y = 0, \qquad \partial\xi_2 / \partial x + \partial\eta_2 / \partial y = 0$$
$$\rho\xi_1 + \eta_2 = 0, \qquad \rho\eta_1 - \xi_2 = 0 \qquad (2.2)$$
$$\zeta^1\xi_1 + \zeta^2\eta_1 - \Gamma^*(2\rho - \rho_{\max} - \rho_{\min}) = 0, \qquad \Gamma^*\rho_* = 0$$

The additional control ρ_* is introduced by Equation

$$(\rho_{\max} - \rho)(\rho - \rho_{\min}) - \rho_*^2 = 0 \qquad (2.3)$$

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* In this paper the notation ξ_1, ... is used for quantities denoted in [5] by $\xi_1 + \varphi_{1y}$, ...

It is convenient to convert Equations (2.2) to another form; let us introduce functions $\omega_1(x,y)$ and $\omega_2(x,y)$ through the relationships

$$\xi_1 = -\partial\omega_1/\partial y, \qquad \xi_2 = -\partial\omega_2/\partial y, \qquad \eta_1 = \partial\omega_1/\partial x, \qquad \eta_2 = \partial\omega_2/\partial x \tag{2.4}$$

The first pair of Equations (2.2) is satisfied identically; the second pair is now written as

$$\rho\partial\omega_1/\partial y = \partial\omega_2/\partial x, \qquad \rho\partial\omega_1/\partial x = -\partial\omega_2/\partial y \tag{2.5}$$

Eliminating ω_2 and ω_1 we subsequently find

$$\frac{\partial}{\partial x}\rho\frac{\partial\omega_1}{\partial x} + \frac{\partial}{\partial y}\rho\frac{\partial\omega_1}{\partial y} = 0, \qquad \frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial\omega_2}{\partial x} + \frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial\omega_2}{\partial y} = 0 \tag{2.6}$$

Problem 2. Function H is equal to

$$H^{(2)} = H^{(1)} - \rho\,[(\zeta^1)^2 + (\zeta^2)^2] \tag{2.7}$$

Equations for Lagrange's multipliers have the form

$$\begin{gathered}\partial\xi_1/\partial x + \partial\eta_1/\partial y = 0, \qquad \partial\xi_2/\partial x + \partial\eta_2/\partial y = 0\\ \rho\xi_1 + \eta_2 + 2\rho\zeta^1 = 0, \qquad \rho\eta_1 - \xi_2 + 2\rho\zeta^2 = 0\\ \zeta^1\xi_1 + \zeta^2\eta_1 - \Gamma^*(2\rho - \rho_{\max} - \rho_{\min}) + (\zeta^1)^2 + (\zeta^2)^2 = 0, \qquad \Gamma^*\rho_* = 0\end{gathered} \tag{2.8}$$

We will introduce functions $\omega_1(x,y)$ and $\omega_2(x,y)$ according to Formulas (2.4). The second pair of Equations (2.8) is now written in the form

$$-\rho\frac{\partial\omega_1}{\partial y} + \frac{\partial\omega_2}{\partial x} + 2\rho\zeta^1 = 0, \qquad \rho\frac{\partial\omega_1}{\partial x} + \frac{\partial\omega_2}{\partial y} + 2\rho\zeta^2 = 0 \tag{2.9}$$

or, if Equations (1.4) (second line) is taken into account

$$\rho\frac{\partial}{\partial y}(\omega_1 + 2z^2) - \frac{\partial\omega_2}{\partial x} = 0, \qquad \rho\frac{\partial}{\partial x}(\omega_1 + 2z^2) + \frac{\partial\omega_2}{\partial y} = 0 \tag{2.10}$$

If ω_2 and ω_1 are eliminated from Equations (2.9), then, taking into account the last equation in the second line of (1.4), we will consequently find

$$\frac{\partial}{\partial x}\rho\frac{\partial\omega_1}{\partial y} + \frac{\partial}{\partial y}\rho\frac{\partial\omega_2}{\partial y} = -\frac{2}{c}\frac{\partial}{\partial x}(vB), \qquad \frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial\omega_2}{\partial x} + \frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial\omega_2}{\partial y} = 0 \tag{2.11}$$

Boundary conditions for functions ω_1 and ω_2 have different forms for different initial boundary problems. These conditions will be spelled out in Section 4.

3. Weierstrass condition. Problem 1. According to the Weierstrass condition the difference

$$\begin{aligned}\Delta H^{(1)} = H^{(1)}(\rho, \zeta^1, \zeta^2) - H^{(1)}(\mathrm{P}, Z^1, Z^2) = -\xi_1(\rho\zeta^1 - \mathrm{P}Z^1) +\\ + \xi_2(\zeta^2 - Z^2) - \eta_1(\rho\zeta^2 - \mathrm{P}Z^2) - \eta_2(\zeta^1 - Z^1)\end{aligned} \tag{3.1}$$

must be positive for all permissible P, Z^1, Z^2; those values of these variables will be admissible which are connected with the optimum values of ρ, ζ^1, ζ^2 through Equations

$$(\rho\zeta^1 - PZ^1)\, x_t + (\rho\zeta^2 - PZ^2)\, y_t = 0, \quad (\zeta^2 - Z^2)\, x_t - (\zeta^1 - Z^1)\, y_t = 0 \tag{3.2}$$

Here x_t and y_t are any real numbers which satisfy the condition $x_t^2 + y_t^2 = 1$ and which have the significance of directional cosines of the tangent to the curve of possible discontinuity of conductivity. Equations (3.2) express continuity of functions z^1 and z^2 along this curve.

We emphasize that controls ρ and P satisfy the last inequality (1.4).

The system of Equations (3.2) permits to eliminate the variables Z^1 and Z^2 from Expression (3.1). After introduction of functions ω_1 and ω_2 and vector $\mathbf{j}(\zeta^1, \zeta^2)$ the Weierstrass condition assumes the form (we omit the calculations)

$$\Delta H^{(1)} = -\frac{\rho - P}{P}\left(\frac{\rho - P}{\rho} j_n \frac{\partial \omega_2}{\partial n} - \mathbf{j}.\,\mathrm{grad}\,\omega_2\right) \geqslant 0 \tag{3.3}$$

Let $\mathbf{n}$ be the direction with direction cosines $(y_t, -x_t)$; inequality (3.3) must be satisfied for arbitrary $\mathbf{n}$.

The structure of the left side in the last inequality shows that the two cases are possible

1) $\quad s = \dfrac{\rho - P}{\rho} > 0, \qquad A = s j_n \dfrac{\partial \omega_2}{\partial n} - \mathbf{j}.\,\mathrm{grad}\,\omega_2 \leqslant 0$

2) $\quad s = \dfrac{\rho - P}{\rho} < 0, \qquad A = s j_n \dfrac{\partial \omega_2}{\partial n} - \mathbf{j}.\,\mathrm{grad}\,\omega_2 \geqslant 0$

C a s e 1 . First of all we note that from the inequality $s > 0$ and the last inequality (1.4) it follows that $\rho = \rho_{max}$. For $\rho = \rho_{max}$ the parameter s varies within the limits

$$0 \leqslant s \leqslant 1 - \frac{\rho_{\min}}{\rho_{\max}} = s_{\max} \leqslant 1 \tag{3.4}$$

Let us assume that $\mathbf{j}.\mathrm{grad}\,\omega_2 < 0$ It is clear that here the condition $A \leqslant 0$ is not fulfilled because a direction $\mathbf{n}$ can always be found for which $j_n \partial\omega_2/\partial n = 0$. The case $\mathbf{j}.\mathrm{grad}\,\omega_2 > 0$ remains. If the direction $\mathbf{n}$ is located within angles aob (Fig. 2), which are bounded by straight lines perpendicular to vectors $\mathbf{j}$ and $\mathrm{grad}\,\omega_2$, then $j_n \partial\omega_2/\partial n < 0$, and the inequality $A \leqslant 0$ is satisfied. If $\mathbf{n}$ is located outside the angles aob, then it is necessary to find the maximum of the function

$$f(\varphi, \psi) = s_{\max} j_n \frac{\partial \omega_2}{\partial n} = s_{\max} j\,|\mathrm{grad}\,\omega_2| \cos\varphi \cos\psi$$

for conditions (see Fig. 2) $\psi = \chi + \varphi$, $\chi = \mathrm{const}$, and it is necessary to require that the corresponding value of A be negative. It is easy to verify that the function $f(\varphi, \chi + \varphi)$ reaches a maximum at $\varphi = -\frac{1}{2}\chi$, i.e. for a direction $\mathbf{n}$, which devides in half the acute angle χ. For this direction

$$f_{\max} = f(-{}^1\!/_2\chi, {}^1\!/_2\chi) = s_{\max} j\,|\mathrm{grad}\,\omega_2| \cos^2({}^1\!/_2\chi)$$

The corresponding value of A is equal to

$$A_{max} = j\,|\text{grad}\,\omega_2|\,[s_{max}\cos^2(^1/_2\chi) - \cos\chi]$$

According to statements above we must have

$$s_{max}\cos^2(^1/_2\chi) - \cos\chi \leqslant 0$$

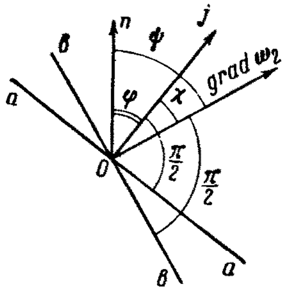

Fig. 2

or

$$(s_{max} - 2)\cos^2(^1/_2\chi) + 1 \leqslant 0$$

From this

$$\chi \leqslant 2\cos^{-1}\frac{1}{\sqrt{2 - s_{max}}} \qquad (3.5)$$

We recall that for

$$\mathbf{j}\cdot\text{grad}\,\omega_2 > 0\ |\chi| \leqslant \pi/2.$$

If one takes into consideration inequality (3.4), it becomes clear that in the case under examination condition (3.5) limits from above the absolute value of the acute angle χ between vectors $\mathbf{j}$ and grad ω_2. The value of the upper limit depends on s_{max}; this limit is equal to $\frac{1}{2}\pi$ for $s_{max} = 0$ ($\rho_{min} = \rho_{max}$) and to 0 for $s_{max} = 1$ ($\rho_{min} = 0$ or $\rho_{max} = \infty$).

C a s e 2. Considerations quite analogous to the ones presented above lead us to the inequality (Fig.3)

$$\chi \geqslant 2\cos^{-1}\frac{1}{\sqrt{2 - s_{min}}} \qquad (3.6)$$

which is fulfilled for the condition

$$\mathbf{j}\cdot\text{grad}\,\omega_2 < 0.$$

Parameter s_{min} is determined by Formula

$$s_{min} = 1 - \frac{\rho_{max}}{\rho_{min}}$$

and in case 2 $\quad s_{min} \leqslant s \leqslant 0$

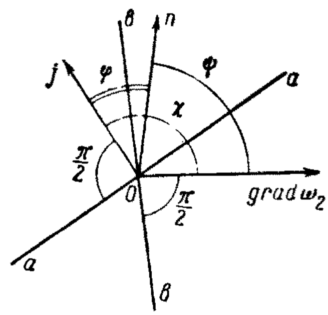

Fig. 3

As we see, the inequality (3.6) limits from below the absolute value of the obtuse angle χ between $\mathbf{j}$ and grad ω_2. The magnitude of the lower limit depends on the value s_{min}; this limit is equal to $\pi/2$ for $s_{min} = 0$ ($\rho_{min} = \rho_{max}$) and to π for $s_{min} = -\infty$ ($\rho_{min} = 0$ or $\rho_{max} = \infty$).

Simultaneously we establish that the condition $A = 0$ can be fulfilled (if one abstains from special cases $\mathbf{j} = 0$ or grad $\omega_2 = 0$) only on individual curves and not in entire regions; this follows from the detrmination of the quantity A which contains the invarient component $s j_n \partial\omega_2/\partial n$.

In addition to this, the scalar product $\mathbf{j}$.grad ω_2 cannot become zero in the optimum regime (with exception of cases mentioned above) because in the opposite case, apparently, the Weierstrass condition would be violated.

Results of material presented above are summarized in the following way.

T h e o r e m . A maximum of functional I with limitations of (1.4) can be achieved for the following optimum controls: (3.7)

$$1)\quad \rho = \rho_{\max} \quad \text{for } \mathbf{j}\cdot\operatorname{grad}\omega_2 > 0,\ \chi \leqslant \cos^{-1} p$$
$$2)\quad \rho = \rho_{\min} \quad \text{for } \mathbf{j}\cdot\operatorname{grad}\omega_2 < 0,\ \chi \geqslant \pi - \cos^{-1} p$$

Parameter p is determined by Formula

$$p = \frac{\rho_{\max} - \rho_{\min}}{\rho_{\max} + \rho_{\min}}$$

P r o b l e m 2. In accordance with the Weierstrass condition, the difference

$$\Delta H^{(2)} = \Delta H^{(1)} - \rho\ [(\zeta^1)^2 + (\zeta^2)^2] + \mathrm{P}\ [(Z^1)^2 + (Z^2)^2] \tag{3.8}$$

must be positive.

Following the reasoning carried out above for Problem 1, we will write directly the analog to inequality (3.3); we obtain

$$\Delta H^{(2)} = -\frac{(\rho - \mathrm{P})^2}{\mathrm{P}\rho} j_n \frac{\partial\omega_2}{\partial n} - \frac{(\rho - \mathrm{P})^2}{\mathrm{P}} j_n^2 + \frac{\rho - \mathrm{P}}{\mathrm{P}} \mathbf{j}\cdot\operatorname{grad}\omega_2 + \frac{\rho - \mathrm{P}}{\rho^*} \rho j^2 \geqslant 0 \tag{3.9}$$

As before, two cases are possible

$$1)\quad s = \frac{\rho - \mathrm{P}}{\rho} > 0, \qquad A^\circ = s j_n \frac{\partial\omega_2}{\partial n} - \mathbf{j}\cdot\operatorname{grad}\ \omega_2 + s\rho j_n^2 - \rho j^2 \leqslant 0$$
$$2)\quad s = \frac{\rho - \mathrm{P}}{\rho} < 0, \qquad A^\circ = s j_n \frac{\partial\omega_2}{\partial n} - \mathbf{j}\cdot\operatorname{grad}\ \omega_2 + s\rho j_n^2 - \rho j^2 \geqslant 0$$

Let us introduce vector $\mu = \operatorname{grad}\omega_2 + \rho\mathbf{j}$. It is not difficult to see that the quantity A° depends on μ exactly in the same way as quantity A depends on $\operatorname{grad}\omega_2$. Therefore, in both cases mentioned we arrive at the same conclusions with respect to optimum controls, as were obtained for Problem 1 and formulated in the theorem. The only difference consists in that in the formulas of the theorem the vector $\operatorname{grad}\omega_2$ should be replaced by μ , and that the angle χ should be given the significance of the angle between vectors $\mathbf{j}$ and $\mu = \operatorname{grad}\omega_2 + \rho\mathbf{j}$. The latter angle, in agreement with the theorem, may be either acute (case 1) or obtuse (case 2). This, however, cannot be said now about the angle between vectors $\mathbf{j}$ and $\operatorname{grad}\omega_2$. This angle may be obtuse both in cases 1 and 2 (but acute only in case 1). The condition indicated can result under certain conditions in nonuniqueness of optimum regimes which are determined by the Weierstrass condition (see Section 4). It is clear that there is nothing paradoxical in this because the Weierstrass condition in itself is a necessary condition for a strong relative minimum. Within the framework of the utilized method the absolute minimum has to be determined by direct computation of values of the functional for relative minima and by comparison of these values among each other.

4. Example. Homogeneous magnetic field B; the velocity depends only on coordinate y (a solution of the problem for the case of constant conductivity was obtained by Vatazhin [1]).

A schematic of the arrangement is shown in Fig.1; boundary conditions which express the constancy of potential z^1 on electrodes, and the disappearance of the normal component of current density ζ^2 on insulators, and also conditions at infinity and Ohm's law for the electrical circuit, have the form

$$z^1(x, \pm\delta) = z_{\pm}^1 = \text{const}, \qquad |x| < \lambda$$

$$z^2(x, \pm\delta)\,|_{x>\lambda} = z_+^2 = \text{const}, \qquad z^2(x, \pm\delta)\,|_{x<-\lambda} = z_-^2 = \text{const}$$

$$z^1(\infty, +\delta) - z^1(\infty, -\delta) = z^1(-\infty, +\delta) - z^1(-\infty, -\delta) = \frac{1}{c} B \int_{-\delta}^{\delta} v dy = \varepsilon\,,$$

$$z^2(\infty, \pm\delta) - z^2(-\infty, \pm\delta) = R^{-1}[z_+^1 - z_-^1] \tag{4.1}$$

Problem 1. Taking into account Equations (2.2) let us write the following expression for the first variation of functional I. The expression was formed by means of Lagrange's multipliers [5] (see Fig.1)

$$\left(\int_E^F + \int_G^H\right)[\eta_1 \delta z^1 + \eta_2 \delta z^2]\, dt - \left(\int_A^B + \int_C^D\right)[\eta_1 \delta z^1 + \eta_2 \delta z^2]\, dt +$$

$$+ \int_F^G [\eta_1 \delta z^1 + \eta_2 \delta z^2]\, dt - \int_B^C [\eta_1 \delta z^1 + \eta_2 \delta z^2]\, dt - \int_H^A [\xi_1 \delta z^1 + \xi_2 \delta z^2]\, dt +$$

$$+ \int_D^E [\xi_1 \delta z^1 + \xi_2 \delta z^2]\, dt - \delta z_+^2 + \delta z_-^2 \tag{4.2}$$

Equating this expression to zero we obtain the boundary conditions for Lagrange's multipliers; it is necessary to take into account in this case that the variations entering here are connected with relationships which were obtained by variation of Equations (4.1). In addition to this, vertical sections HA and DE (Fig.1) should be moved to infinity. We obtain

on electrodes (FG and BC)

$$\eta_2 = 0 \tag{4.3}$$

on insulators (EF, GH, AB and CD)

$$\eta_1 = 0 \tag{4.4}$$

at infinity

$$\int_H^A \xi_1\, dt = \int_D^E \xi_1\, dt = 0, \qquad \int_H^A \xi_2\, dt = \int_D^E \xi_2\, dt = 0 \tag{4.5}$$

Terms remaining in (4.2) form Equation

$$\left[\int_E^F \eta_2\,dt - \int_C^D \eta_2\,dt - 1\right]\delta z_+^{\,2} + \left[\int_G^H \eta_2\,dt - \int_A^B \eta_2\,dt + 1\right]\delta z_-^{\,2} + $$
$$+ \int_F^G \eta_1\,dt\;\delta z_+^{\,1} - \int_B^C \eta_1\,dt\;\delta z_-^{\,2} = 0 \qquad (4.6)$$

Variations entering here are connected through the relationship

$$\delta z_+^{\,1} - \delta z_-^{\,1} = R\,[\delta z_+^{\,2} - \delta z_-^{\,2}]$$

Eliminating variation $\partial z_+^{\,1}$ from (4.6) by means of this equation, we arrive at a relationship in which the variations may be considered already independent and the corresponding coefficients may be equated to zero. We obtain

$$\int_F^G \eta_1\,dt = \int_B^C \eta_1\,dt$$
$$\int_E^F \eta_2\,dt - \int_C^{D'} \eta_2\,dt - 1 = -R\int_F^G \eta_1\,dt$$
$$\int_G^H \eta_2\,dt - \int_A^B \eta_2\,dt + 1 = R\int_F^G \eta_1\,dt \qquad (4.7)$$

We will write the obtained relationships utilizing functions ω_1 and ω_2 introduced above. We shall have

on the electrodes

$$\omega_2(x, \pm\delta) = \omega_{2\pm} = \text{const}, \qquad \partial\omega_1/\partial y = 0 \qquad (4.8)$$

on the insulators

$$\omega_1(x, \pm\delta)\,\big|_{x>\lambda} = \omega_{1+} = \text{const}, \; \omega_1(x, \pm\delta)\,\big|_{x<-\lambda} = \omega_{1-} = \text{const}$$
$$\partial\omega_2/\partial y = 0 \qquad (4.9)$$

at infinity

$$\omega_1(\infty, \delta) = \omega_1(\infty, -\delta), \qquad \omega_1(-\infty, \delta) = \omega_1(-\infty, -\delta)$$
$$\omega_2(\infty, \delta) = \omega_2(\infty, -\delta), \qquad \omega_2(-\infty, \delta) = \omega_2(-\infty, -\delta) \qquad (4.10)$$

In addition to this

$$\omega_{2+} - \omega_{2-} + 1 = R\,[\omega_{1+} - \omega_{1-}] \qquad (4.11)$$

If we now introduce the function u, which is related to z^1 by Equation

$$u = z^1 - \frac{B}{c}\int_0^y v\,dy \qquad (4.12)$$

then Equations (1.4) will be rewritten in an equivalent form

$$\frac{\partial z^2}{\partial y} = \frac{1}{\rho}\frac{\partial u}{\partial x}, \qquad \frac{\partial z^2}{\partial x} = -\frac{1}{\rho}\frac{\partial u}{\partial y} \qquad (4.13)$$

and vector $\mathbf{j}$ becomes

$$\mathbf{j} = -\rho^{-1}\,\mathrm{grad}\,u$$

Equations (4.13) will coincide with Equations (2.5) if the latter equations ω_2 is replaced by u and ω_1 by z^2 . A comparison of boundary conditions (4.1) and (4.8) to (4.11) shows that for any function $\rho(x,y)$ the relationships

$$z^2 = \varepsilon\omega_1, \qquad u = \varepsilon\omega_2 \tag{4.15}$$

apply.

Equations (4.14) and (4.15) show that vectors $\mathbf{j}$ and grad ω_2 are antiparallel everywhere $(\chi = \pi)$. Remembering Weierstrass criterion for problem 1 (theorem), we conclude that for the optimum regime $\rho = \rho_{\min}$ everywhere, a result which is in complete agreement with considerations of a physical nature.

We note that utilizing solution [1], we would have also arrived at this conclusion. However, the maximum of functional I would have been determined with respect to a class of functions of the equation which assume the same constant value everywhere, while the result obtained by the general method [5] applies to a broader class of sectionally continuous functions of two independent variables.

P r o b l e m 2 In this problem boundary conditions (4.8) to (4.10) are preserved for Lagrange's multipliers; instead of condition (4.11) the following equality is maintained:

$$\omega_{2+} - \omega_{2-} + 2IR = R\,[(\omega_{1+} + 2z_+^2) - (\omega_{1-} + 2z_-^2)] \tag{4.16}$$

As before, we find that functions u (see (4.12)) and ω_2 are related by Equation

$$u = \frac{\varepsilon}{2IR}\,\omega_2$$

The Weierstrass criterion now assumes the form

$$\Delta H^{(2)} = \frac{\rho - P}{P}\,\rho\left(\frac{2IR}{\varepsilon} - 1\right)\left[\frac{\rho - P}{\rho}\,j_n^2 - j^2\right] \geqslant 0$$

From this it is not difficult to draw conclusions with regard to the possibility of the optimum regimes

1) $2IR\,/\,\varepsilon < 1, \quad \rho = \rho_{\max}$

2) $2IR\,/\,\varepsilon > 1, \quad \rho = \rho_{\min}$

3) $2IR\,/\,\varepsilon = 1,$ special case

For ρ = const the expression $2IR/\varepsilon$ is easy to compute, it is equal to [1]

$$\frac{2IR}{\varepsilon} = \frac{2R\alpha}{2\rho + R\alpha}$$

parameter α is given by the relation

$$\alpha = \left[\frac{K(k)}{K(k')}\right]^{-1}, \qquad k = \exp\left(-\frac{\lambda\pi}{\delta}\right), \quad k' = \sqrt{1-k^2}$$

Here $K(k)$ is a complete elliptical integral of the first kind.

Inequalities which apply to regimes 1 and 2 take correspondingly the form

$$R < 2\rho_{max}/\alpha, \qquad R > 2\rho_{min}/\alpha$$

From this it follows that for $R < 2\rho_{min}/\alpha$ only the control $\rho = \rho_{max}$ is possible , for $R > 2\rho_{max}/\alpha$ only the control $\rho = \rho_{min}$ is possible. If parameter R is included in the interval $(2\rho_{min}/\alpha, 2\rho_{max}/\alpha)$ then the Weierstrass condition permits both controls ρ_{max} and ρ_{min}. A similar possibility was already discussed at the end of Section 3. It remains to point out the citerion for the determination of an absolute minimum. It is easy to verify, using expression [1] for function Q for ρ = const

$$Q = \frac{\varepsilon^2}{[2(\rho/\alpha)+R]^2}\frac{2\rho}{\alpha}$$

that the absolute minimum is reached when

$$\rho = \rho_{max}, \quad \text{when } 2\alpha^{-1}\rho_{min} < R < 2\alpha^{-1}\sqrt{\rho_{max}\rho_{min}}$$

$$\rho = \rho_{min}, \quad \text{when } 2\alpha^{-1}\sqrt{\rho_{max}\rho_{min}} < R < 2\alpha^{-1}\rho_{max}$$

As far as the special regime is concerned, it should be disregarded since already with respect to the class of functions of the equation, which assume a constant value everywhere, this regime corresponds to a maximum and not a minimum of functional I . This is confirmed by direct computation. The Weierstrass criterion is fulfilled in this case in the weak sense.

In conclusion we note that all deductions made for Problem 2 could have been obtained from the corresponding statements for Problem 1 with the aid of Equation

$$Q = I\varepsilon - I^2 R$$

which holds for the case of homogeneous magnetic field B .

Here the results were obtained directly for the purpose of illustrating practical examples which are typical for application of the general method [5].

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